Pyramid Puzzle

Task 101 ... Years 2 - 10

Summary

The 3D spatial challenge that is the beginning of the task can be tried by students of any age. Solving it suggests a number pattern in the number of spheres at each level and that too can be accessed by quite young children. Seeing the pattern leads to predicting based on it and that's where things start to get more difficult. It's not so much predicting the number of balls in any given layer of a growing pyramid that is challenging, rather, it is predicting the total number of balls needed to make a pyramid of that size. Further, asking the question:

- *Can I check it another way?*

could lead to using Proof by Mathematical Induction which is Year 12 content.

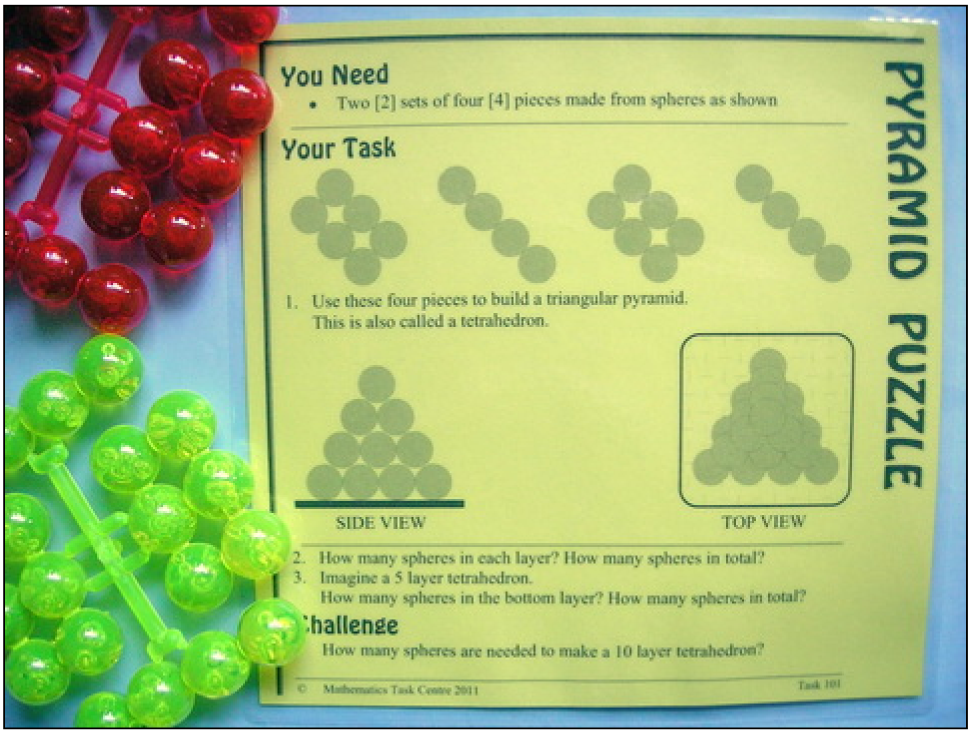
This cameo has a From The Classroom section which derives from a teacher connecting two pieces of mathematics, one of which includes the work of two Year 10 students, and coming up with a new hypothesis. It also confirms how deeply embedded the process of working like a mathematician is in this school.

Materials

- 4 sets of spheres for each person.
  One set is 2 pieces 1x4 and 2 pieces 2x3

Content

- 3D (and 2D) spatial perception
- recognising & interpreting number patterns
- Natural Numbers
- summing Natural Numbers
- Triangle Numbers
- summing Triangle Numbers
- Square Numbers
- summing Square Numbers
- proof by Mathematical Induction

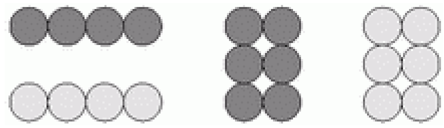


Iceberg

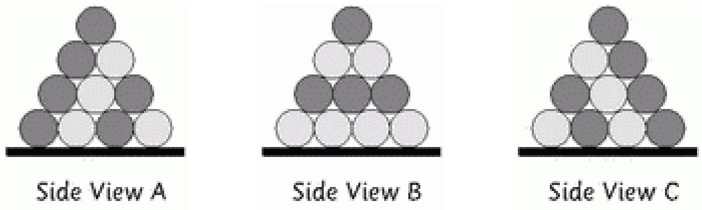
A task is the tip of a learning iceberg. There is always more to a task than is recorded on the card.

Hint: Give a paper plate to each student along with the puzzle. This will (a) provide a non-slip surface and (b) make it almost impossible for the longer pieces to roll onto a hard floor and perhaps snap.

If these are the pieces:



The solution can be derived from these diagrams:



Or, for those who prefer an oral explanation, Side View A gives a clue:

1. Place a 4 piece so that it points away from you. This item has a depth of 4.
2. On its left, stack a piece with a depth of 3.
3. On the left of these two, stack a piece with a depth of 2.
4. On the left of these three, stack a piece with a depth of 1.

Never give students the solution to this puzzle. Someone will work it out one day. It falls into the category of easy-when-you-know-how. Encourage those who do find the solution to keep it to themselves, or, if someone else asks for help, they may tell *provided* they keep their hands behind their back.

- *What happens if a 4 layer tetrahedron is dissected another way? Could that create a worthwhile new puzzle?*
- *Is it possible to create such a dissection using only four 2D pieces?*

Using polystyrene balls and toothpicks students could explore creating such a puzzle. (Send a photo if your students create Pyramid Puzzle B, C , D...)

Students can work out the answers to Question 2 either by counting as they manipulate their pyramid, or by using the clues in the Side View on the card.

Layer	# Spheres	Spheres So Far	Total
1	1	1	1
2	3	1 + 3	4
3	6	1 + 3 + 6	10
4	10	1 + 3 + 6 + 10	20

All faces of a regular tetrahedron are identical, so the Side View is also the Base View. This realisation helps work out a 5 layer tetrahedron. Continuing from above gives:

5	15	1 + 3 + 6 + 10 + 15	35

And continuing the pattern to a 10 layer tetrahedron gives:

10	55	1 + 3 + ... + 45 + 55	220

Extension A

One direction for the iceberg of this task begins with the question:

- *If I tell you any layer of the tetrahedron, can you tell me how many spheres there are in that layer?*

In the Nth layer of the tetrahedron the number of spheres is counted by the Nth Triangle Number - not surprising really since each layer is a progressively larger triangle. In its turn, the Nth Triangle number is found by adding the Natural Numbers from 1 to N. Again, this relates to the way the triangle is formed - first one sphere, then two, then three and so on to build the base layer.

There are several ways to calculate:

Sum of Natural Numbers = $\boldsymbol{S}_N = 1 + 2 + 3 + ... + (N-2) + (N-1) + N$

so, depending on your students, you might encourage the mathematician's question:

- *Can I check it another way?*

And if your class is Year 12, this is an opportunity to introduce Proof by Mathematical Induction as one of those other ways.

A more challenging iceberg question is:

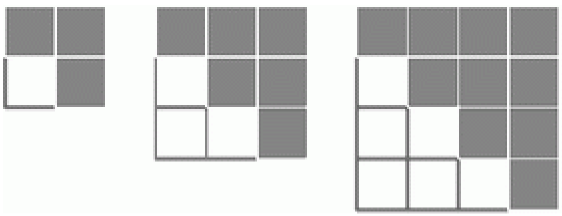
- *If I tell you the size of the tetrahedron, can you tell me how many spheres it takes in total to make it?*

From the table above we can see that to do this we need to be able to sum the Triangle Numbers.

Sum of Triangle Numbers = $\boldsymbol{S}_T = 1 + 3 + 6 + ... + {}^{N(N+1)}/_{2}$

where N is the number of layers in the tetrahedron.

The clue to achieving this calculation is in recognising that each pair of consecutive Triangle Numbers makes a square:



It seems that to find the sum of the Triangle Numbers it will first be necessary to sum the Square Numbers. So another new challenge arises. Not necessarily Year 12 stuff since Greek mathematicians were able to derive a formula using spatial reasoning, but definitely open again to checking the solution by Mathematical Induction.

Extension B

Another direction for the iceberg begins with the question:

- *What happens if we build square-based pyramids with spheres?*

Explore the number of spheres in each layer and the total number of spheres needed to make the pyramid to a given layer.

Whole Class Investigation

Tasks are an invitation for two students to work like a mathematician. Tasks can also be modified to become whole class

It is sometimes remarkable how much more mathematics students see when their hands are holding the object of mathematical inquiry. You may be tempted to generate a *investigations which model how a mathematician works.*

whole class investigation of this problem based on it being used by various groups in previous task sessions, or perhaps even from drawings, but there is no real substitute for a class set of Pyramid Puzzles. (Unless, perhaps, you have access to a snooker triangle and enough balls to build a pyramid above the balls usually supplied.) Some craft shops supply polystyrene spheres. With a bit of glue (and perhaps some bamboo skewers through the balls to make them stronger) it's easy to make the 4 piece and the 6 piece. Perhaps students can help make enough pieces for a class set.

The general outline of the whole class investigation is above. More detail, including formulas for the sums of Natural Numbers, Square Numbers and Triangle Numbers, and proofs by Mathematical Induction can be found in the companion Maths300 lesson.

For more ideas and discussion about this investigation, open a new browser tab (or page) and visit Maths300 Lesson 138, *Pyramid Puzzle & Other Algebra Investigations*.

Is it in Maths With Attitude?

Maths With Attitude is a set of hands-on learning kits available from Years 3-10 which structure the use of tasks and whole class investigations into a week by week planner.

The *Pyramid Puzzle* task is an integral part of:

- MWA *Pattern & Algebra* Years 9 & 10

The *Pyramid Puzzle* lesson is an integral part of::

- MWA *Pattern & Algebra* Years 9 & 10

From The Classroom

MacKillop College, Swan Hill

Damian Howison

Year 10^{+}

G'day Doug,
I have just this moment had a small epiphany, so I need to write it down before it vanishes. It concerns the Pyramid Puzzle and the Twelve Days of Christmas.

The gifts given according to the song grow from day to day like the pyramid, each day's gifts being a new layer added to the bottom of the growing pyramid. I'm sure this is nothing new to you, but I only just realised that the pieces of glued spheres used for the puzzle represent the numbers of gifts accumulated according to which type of gift it is.

For example, the actual pieces (arrays of spheres) that come with Pyramid Puzzle (representing the first four days of the song) are:

- 4 partridges - 4 days x 1 partridge
- 6 turtle doves - 3 days x 2 doves
- 6 french hens - 2 days x 3 hens
- 4 colly birds - 1 day x 4 birds

So could the 5-layer pyramid be made of the 5 pieces based on the song?

- 5 partridges - 5 days x 1 partridge
- 8 turtle doves - 4 days x 2 doves
- 9 french hens - 3 days x 3 hens
- 8 colly birds - 2 days x 4 birds
- 5 gold rings - 1 day x 5 rings

If so, in this way you can see the pyramid as:

a. an accumulation of days with triangular numbers of gifts, and
b. an accumulation of gifts with rectangular numbers for each type of gift.

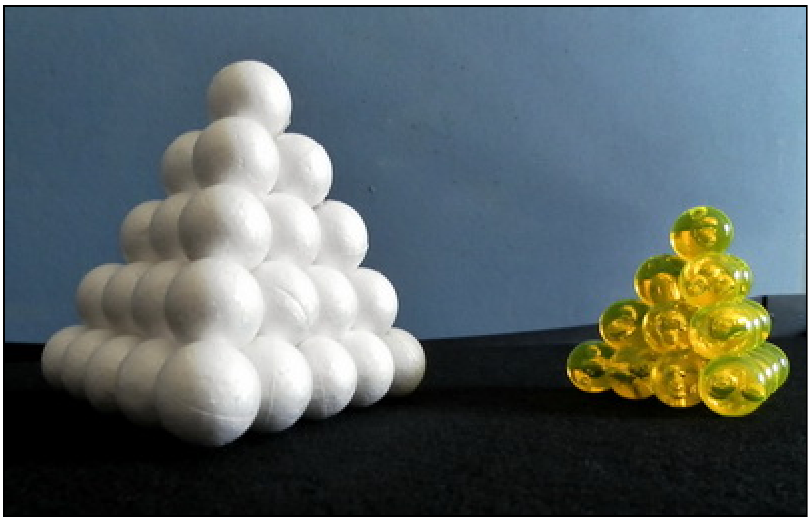
I've known those patterns for some time but never connected the second pattern to the song itself. The Pyramid Puzzle always seems to have another surprise in store!

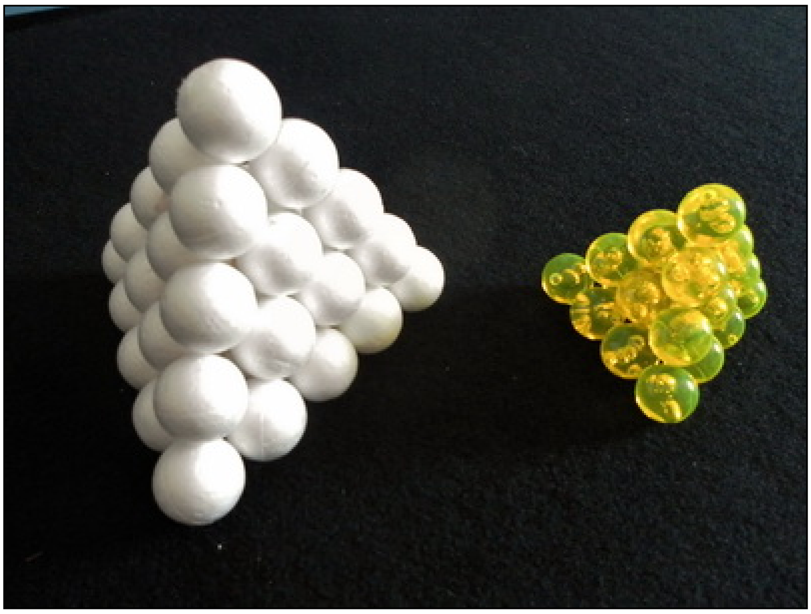
Now Damian, it's one thing to see the pattern, and even recognise that the total of spheres is the number needed to build a Size 5 pyramid, but do you mean that these five pieces will actually fit together to make a Size 5 pyramid?!?

Actually that's exactly what he did mean, as explained in a following email with photos attached:

Morning Doug,
Actually I had a pair of year 10 students try out the 5-layer pyramid last year as part of their investigation in our algebra replacement unit. So I already have the polystyrene pieces! I've photographed them just now.

I had intended to send some photos to you and was hoping that some students pick up on it again this year because the task was a very popular choice in the kit this year. But none of the groups wanted to go that way with their work. Anyway, here are some photos from some pieces made by a pair of Hannahs.





Actually I just traced back to last year's reports and found their work. One of the reports also contains some good photos.
R+,
Damian Howison

So the hypothesis is that for a Size N pyramid (tetrahedron) a construction puzzle can be created by choosing pieces using the 12 Days of Christmas song up to the Nth day. The pieces are then also placed according to the song beginning with the Nth day, ie: 1 x N spheres (x = 'rows of'). The proposition makes sense because:

- 1 x N has (N - 1) junctions between the spheres in which the 2 x (N - 1) piece can rest.
- 2 x (N - 1) makes the structure one row higher and provides (N - 2) junctions between spheres in which the 3 x (N - 2) piece can rest.
- 3 x (N - 2) makes the structure one row higher and provides (N - 3) junctions between spheres in which the 4 x (N - 3) piece can rest.
- ...
- (N - 1) x 2 makes the structure one row higher and provides 1 junction between spheres in which the N x 1 piece can rest.
- N x 1 completes the structure.

Probably not the strictest proof because it depends on visualisation, but it 'feels right'. We would be very happy to publish a stricter version - or a disproof if the reasoning is incorrect. Over to you.

Follow this link to Task Centre Home page.